Structuralism

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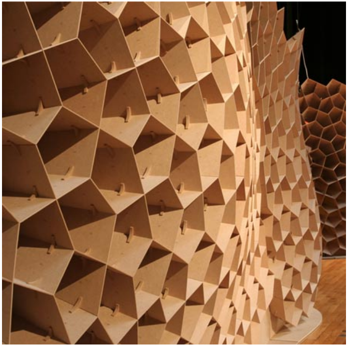
Introduction to Philosophy of Mathematics

Structuralism, a first approach



- mathematics as the science of structure
- usually: realists in truth-value, not necessarily in ontology (Benacerraf and Hellman don't presuppose existence of mathematical objects, Resnik and Shapiro do, "after a fashion")

The underlying idea

- ontological platonism: independent existence of object (i.e. whether an object exists doesn't depend on the existence of others)
- vigorously rejected by structuralists: for them, the essence e.g. of a natural number is its relations to other natural numbers

⇒ "subject-matter of arithmetic is a single abstract structure, the pattern common to any infinite collection of objects that has a successor relation, a unique initial object, and satisfies the induction principle." (Shapiro, 258)

- real analysis: study of "pattern of any complete real closed field", etc
- So let's introduce the central terms for the following discussion...

Systems and structures

Characterization (System)

"Define a system to be a collection of objects with certain relations among them." (259)

- comment: these are concrete collections of objects with concrete relations exemplified by n-tuples of objects
- examples: chess configuration is system of chess pieces with spatial and 'possible move' relations; a government is a system of people with certain supervisory and co-worker relations (and possibly others)

Characterization (Structure)

"Define a pattern or structure to be the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system." (ibid.)

Structure, set-theoretically

For more details, see my handout ‘Structure’ available at

https://wuthrich.net/teaching/2010_246/246HandoutStructureMath_2010.pdf

Note:

This raises the problem of consistently grounding sets prior to structures; may perhaps be avoided by replacing set-theoretic concepts by unregimented concepts of ‘collections’ etc.

Definition (Structure, set-theoretically)

A (relational) structure $\mathcal{S}$ is an isomorphism class of ordered pairs $\langle \mathcal{O}, \mathcal{R} \rangle$ which consists of a non-empty set of relations $\mathcal{R}$ (‘ideology’) as well as a non-empty set of relata $\mathcal{O}$ (‘ontology’), the domain of $\mathcal{S}$ or dom($\mathcal{S}$).

Definition (Isomorphism class)

An isomorphism class is a set of all objects (in some background domain) which are pairwise isomorphic. Two objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic just in case there exists an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Definition (Relation)

An n-ary relation defined on sets $X_1, ..., X_n$ is a set of ordered n-tuples $\langle x_1, ..., x_n \rangle$, where $x_i \in X_i$ for all $i = 1, ..., n$. Thus, an n-ary relation on sets $X_1, ..., X_n$ is just a subset of the Cartesian product $X_1 \times \cdots \times X_n$ of these sets.

- For example: A binary relation B on a domain D is a set of ordered pairs $\langle x, y \rangle$ with $x, y \in D$, and thus a subset of the Cartesian product of D with itself.

Definition (Cartesian product)

The Cartesian product $X_1 \times \cdots \times X_n$ of $X_1, ..., X_n$ can be defined as the set of all ordered n-tuples $\langle x_1, ..., x_n \rangle$ such that for all $i = 1, ..., n$, $x_i \in X_i$.

Homomorphisms and isomorphisms

Definition (Homomorphism)

A homomorphism is a structure-preserving map, i.e., a map which preserves relations (and functions). Thus, a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a map $h : dom(\mathcal{A}) \rightarrow dom(\mathcal{B})$ such that for any n-ary relation R and any elements $a_1, ..., a_n \in dom(\mathcal{A})$, if $\langle a_1, ..., a_n \rangle \in R$, then $\langle h(a_1), ..., h(a_n) \rangle \in R$.

Definition (Isomorphism)

A bijective map $f : dom(\mathcal{A}) \rightarrow dom(\mathcal{B})$ is called an isomorphism just in case both f and its inverse f^{-1} are homomorphisms.

Michael Resnik

In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we only have structures. The objects of mathematics, that is, the entities, which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure. (Resnik 1981, cited after Shapiro, 259)

Structuralism in mathematics

- identity of ‘objects’ in structure exhausted by their position in the structural complex, by their ‘relational profile’
- they have no ‘internal’ or further structures

$\Rightarrow$ a real number is nothing but a position in the real number structure

- structure is to structured, as patterns is to patterned, as universal is to subsumed particular, as type is to token

Two questions concerning ontology

1. What is the status of structures themselves?
2. What is the status of individual mathematical objects (= places within structures)?

- take clues from philosophical literature on universals: ante rem structuralism (à la Plato) vs. in re structuralism (à la Aristotle)

Ante rem and in re

Characterization (Ante rem structuralism)

Ante rem structuralism takes the 'one-over-many' to be ontologically prior to the 'many', i.e., structures do not ontologically depend on the systems that instantiate them. The subject-matter of mathematics are thus abstract, platonic structures.

Characterization (In re structuralism)

In re structuralism takes the view that structures ontologically depend on their instances, i.e., on the particular systems that exemplify them. Thus, ontologically speaking, the 'many' is prior to the 'one-over-many'. The subject-matter of mathematics is the structure that is common to these fundamental systems.

- First question above thus asks "whether, and in what sense, structures themselves exist independently of the systems of objects that exemplify them." (263)

Ante rem structures, and objects

- ante rem structuralism: structure prior to places it contains, just as governmental organization is prior to the offices that constitute it
- solves one problem of realism in ontology taken seriously by some platonists: Frege's Caesar problem
- concrete case: reduction of arithmetic to set theory, or what are numbers?
  1. Zermelo numbers: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$
  2. von Neumann numbers: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$

$\Rightarrow$ problem for reductionist realist in ontology: which sets are the natural numbers?

- ante rem structuralist: this question needs no answer—numbers are just positions in the abstract structure shared by both systems

- This solves the problem since the identity of mathematical objects is exhausted by their position in a structural complex.

$\Rightarrow$ ontological relativity, since mathematical objects are thus not independent of the structures in which they feature

- furthermore: identify positions in $\mathbb{N}$ with their counterparts in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$\Rightarrow$ sometimes it makes sense to identify positions in different structures

Two perspectives on objects

1. places-are-offices perspective: positions of structure are treated in terms of the objects occupying that positions
   - positions are treated "more like properties than objects" (e.g.: the goalkeeper today was a midfielder yesterday)
   - presupposes background ontology of people, sets, small, moveable objects etc
2. places-are-objects perspective: places of structure treated as objects in their own right
   - statements are about structure qua structure, independently of any exemplification (e.g.: the Vice-President is President of the Senate)
   - ante rem structuralist: mathematical objects are bona fide objects in this sense

Note ante rem structuralism: distinction between office and office-holder is relative (at least in maths), i.e., what is object from one perspective may be position in a structure from another

Structuralism without structures

- statements from the places-are-objects perspective entail generalizations over all systems in the pertinent isomorphism class, i.e., they apply to all objects occupying a given position in any system in that isomorphism class (everyone who is Vice-President is also President of the Senate)

⇒ in re structuralist may hold that statements from places-are-objects perspective are really just handy short form of the corresponding generalizations over all systems in the isomorphism class

⇒ in that view, places-are-objects view superfluous

⇒ eliminative structuralism: "paraphrase[] places-are-objects statements in terms of the places-are-offices perspective." (271)

⇒ requires robust background ontology (to underwrite Cantor's Heaven)

Two ways of dealing with infinite ontological demands
(1) ontological eliminative structuralism

- postulate the existence of sufficiently many abstract objects for your background ontology to make room for the hierarchy of Sets
- This background ontology cannot be interpreted along structuralist lines: "If the set-theoretic hierarchy is the background, then set theory is not, after all, the theory of a particular structure." (273)

$\Rightarrow$ grounding problem (but can arguably be solved)

(2) modal eliminative structuralism

- more in a nominalist vein: cash out in terms of possible structures rather than (actual) structures
- mathematical statements understood as statements inside the scope of a modal operator

⇒ don't need rich background ontology, but rich modal background ontology

⇒ puzzle: how to keep arithmetic, real analysis, etc, from being vacuous without stipulating a system that exemplifies the relevant structure(s)

- also: what is the nature of the invoked possibility? physical? metaphysical? logical?

(3) fictionalist eliminative structuralism

Hartry Field, *Realism, Mathematics, and Modality*, Oxford: Blackwell, 1989.

- fictionalism in mathematics: mathematical theories are like fictional stories, and their objects like fictional characters (first articulated in introductory chapter of Field 1989)
- transposed: eliminative structuralism which supplies background ontology by stipulating a rich ontology of fictional entities
- Problems: What are fictional entities? Is Caesar problem really solved? Given its kinship with formalism, does it also suffer from the latter's problems?